

CH 5: Failures Resulting from Static Loading

A static load is a stationary (*constant magnitude, location and direction*) force or couple (moment or torque) applied to a member.

Failure in machine design means that a part become permanently distorted (*i.e. yielded*) thus its function is compromised. Therefore, the failure strength we refer to is the yield strength (or if the material does not yield such as some brittle materials failure will mean fracture “ultimate strength”).

- When a material is subjected to uniaxial stress, it will fail when the stress reaches the failure strength.
 - *What about if it is subjected to biaxial or triaxial stress, when would it fail?*
 - *Is there a difference in the failure mechanisms of ductile and brittle materials?*
 - *Should stress concentrations be considered?*

Static Strength

When designing a machine element, the best way to determine its strength is to test the part under exactly the same loading conditions the part will experience in service. (*of course many tests are needed to account for variability*). However, performing such testing is very costly and cannot be justified unless the part is used for a critical application or if the part is produced in large quantities.

Thus, in most cases it is necessary to design using the published material properties (*yield strength, ultimate strength, percentage elongation, etc.*) which are obtained from testing under uniaxial stress.

How to deal with 2D and 3D states of stress?

Use failure theories

Stress concentration

As seen previously in chapter 3, stress concentration is a highly localized effect.

Would stress concentrations cause failure of a part?

- If the material is ductile (*and the load is static*), the design load might cause local yielding in locations having high stress concentrations and the stress is redistributed in the surrounding area thus the load can be carried without causing failure of the part. Thus, for ductile materials, stress concentrations are not considered ($K_t = 1$).
- Brittle materials, in general, do not yield before fracture thus with the presence of stress concentration, the stress will continue to increase until it cause failure of the part. Therefore, for brittle materials stress concentrations must be considered.
 - An exception to this rule for brittle materials is for materials containing micro discontinuities (bubbles, graphite flakes, etc.) such as cast iron where those act as cracks thus causing higher stress concentrations than those caused by holes or fillets. Therefore, stress concentrations should not be considered for materials such as cast iron.

Failure Theories (Failure Criteria)

Failure theories are used to predict if failure would occur under any given state of stress.

- There are several failure theories that are used in practice.
- Because brittle and ductile materials have different failure mechanisms, different failure theories are used for each.
 - For ductile materials (*yield criteria*):
 - Maximum Shear Stress (**MSS**) - *also known as “Tresca” Criteria*.
 - Distortion Energy (**DE**) - *also known as “Von Mises” Criteria*.
 - Ductile Coulomb-Mohr (**DCM**).
 - For brittle materials (*fracture criteria*):
 - Maximum Normal Stress (**MNS**).
 - Brittle Coulomb-Mohr (**BCM**) and its modifications.

Maximum Shear Stress Theory (for Ductile Materials)

The *MSS* theory came from the observation that for ductile materials during tension test, the slip lines (*which occur after yielding*) and fracture surfaces occur at 45° angle, which is the angle of the maximum shear stress.

- The *MSS* theory predicts that yielding occurs when the maximum shear stress (*for any given state of stress*) is equal to or larger than the shear stress at yielding in a tension-test specimen of the same material.

- During tension test the stress is $\sigma = P/A$ and the maximum shear stress occurs at 45° angle and it is equal to $\tau_{max} = \sigma/2$ and at yielding the maximum shear stress is:

$$\tau_{max} = \frac{S_y}{2}$$

- For a general state of stress, the principal stresses are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and thus the maximum shear stress is:

$$\tau_{max} = (\sigma_1 - \sigma_3)/2$$

- Thus, according to the *MSS* theory, yielding occurs when:

$$\rightarrow \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \quad \text{or} \quad \boxed{\sigma_1 - \sigma_3 \geq S_y}$$

- For plane stress condition (*one of the principal stress is zero*) three cases can be considered:

- Case 1:** $\sigma_A \geq \sigma_B \geq 0$, thus $\sigma_1 = \sigma_A$ & $\sigma_3 = 0$ and yielding occurs when

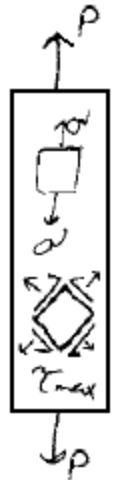
$$\boxed{\sigma_A \geq S_y}$$

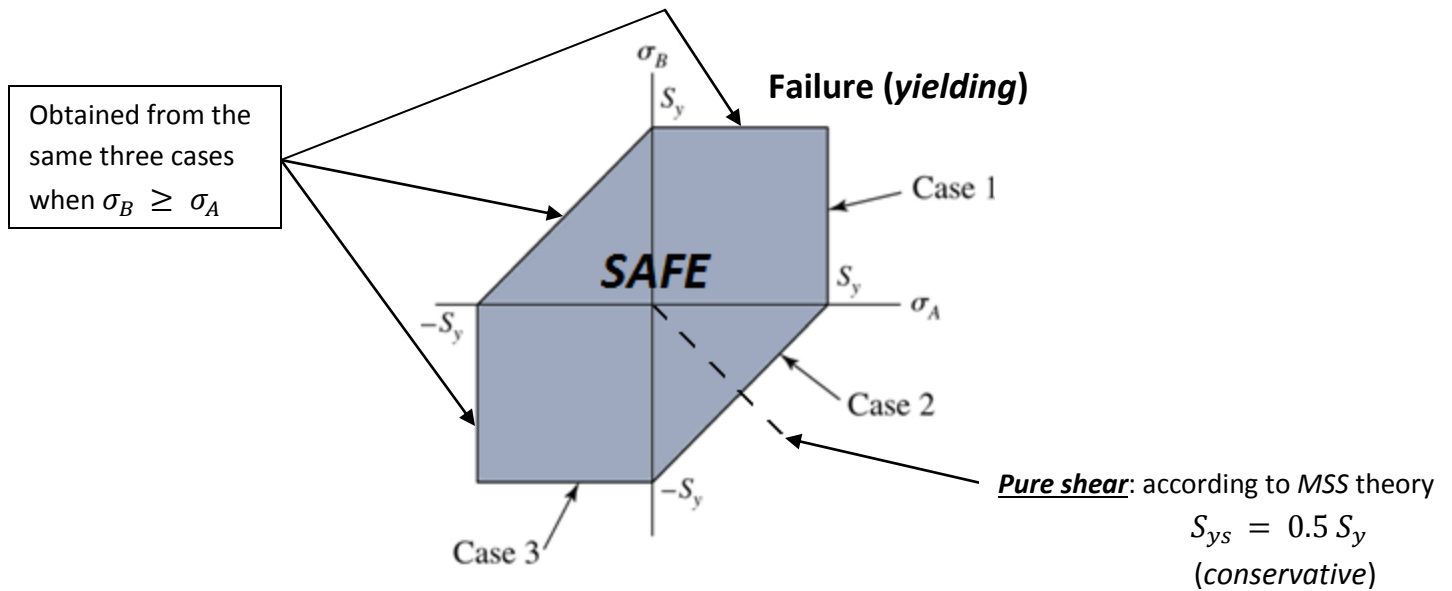
- Case 2:** $\sigma_A \geq 0 \geq \sigma_B$, thus $\sigma_1 = \sigma_A$ & $\sigma_3 = \sigma_B$ and yielding occurs when

$$\boxed{\sigma_A - \sigma_B \geq S_y}$$

- Case 3:** $0 \geq \sigma_A \geq \sigma_B$, thus $\sigma_1 = 0$ & $\sigma_3 = \sigma_B$ and yielding occurs when

$$\boxed{\sigma_B \leq -S_y}$$



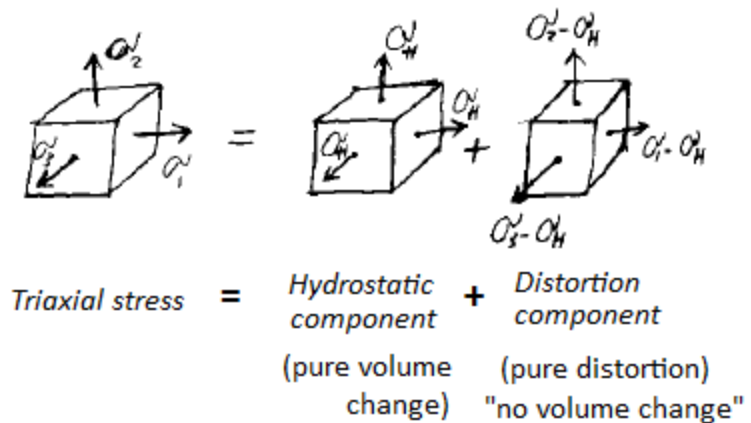


Distortion Energy Theory (for Ductile Materials)

The *DE* theory originated from the observation that ductile materials subjected to hydrostatic stress ($\sigma_1 = \sigma_2 = \sigma_3$) can withstand stress much higher than the yield strength value obtained from tensile test, without yielding. This led to the conclusion that yielding is related to the angular distortion.

- The *DE* theory predicts that yielding occurs when the distortion strain energy (per unit volume) reaches or exceeds the distortion strain energy (per unit volume) for yielding in simple tension test specimen of the same material.

- To develop the theory, consider an element subjected to triaxial state of stress ($\sigma_1 > \sigma_2 > \sigma_3$), the state of stress can be divided into hydrostatic and distortion components where,



$$\sigma_H = \sigma_{Avg} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad (1)$$

- The strain energy (per unit volume) for the element subjected to triaxial stress is:

$$U = \frac{1}{2} (\epsilon_1 \sigma_1 + \epsilon_2 \sigma_2 + \epsilon_3 \sigma_3)$$

- Using the generalized Hook's law for triaxial state of stress we can write:

Total strain energy of the triaxial state of stress

$$U = \frac{1}{2E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (2)$$

- The strain energy due to the hydrostatic stress components (*producing pure volume change*) is obtained by substituting σ_H for $\sigma_1, \sigma_2, \sigma_3$ in equation (2):

Strain energy of the hydrostatic stress component

$$U_V = \frac{3\sigma_H^2}{2E} (1 - 2\nu) \quad (3)$$

- Substituting equation (1) in equation (3) we get:

$$U_V = \frac{1 - 2\nu}{6E} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1] \quad (4)$$

- Thus, the distortion strain energy is obtained by subtracting equation (4) from equation (2):

$$U_d = U - U_V = \frac{1 + \nu}{3E} \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right] \quad (5)$$

➤ Note that the distortion energy will be zero if ($\sigma_1 = \sigma_2 = \sigma_3$)

- For a simple tensile test, at yielding: $\sigma_1 = S_y$ & $\sigma_2 = \sigma_3 = 0$. Thus, from equation (5) the distortion energy is:

Distortion energy at yielding during tensile test

$$U_d = \frac{1 + \nu}{3E} S_y^2 \quad (6)$$

- Thus, according to *DE theory*, yielding occurs when Equation (5) \geq equation (6), which gives:

The DE theory

$$\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \geq S_y$$

- The left-hand-term is known as the Von Mises stress (σ') which is an equivalent, or effective stress representation for any general state of stress.

Von Mises stress

$$\sigma' = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

- Thus, the *DE* theory can be written as:

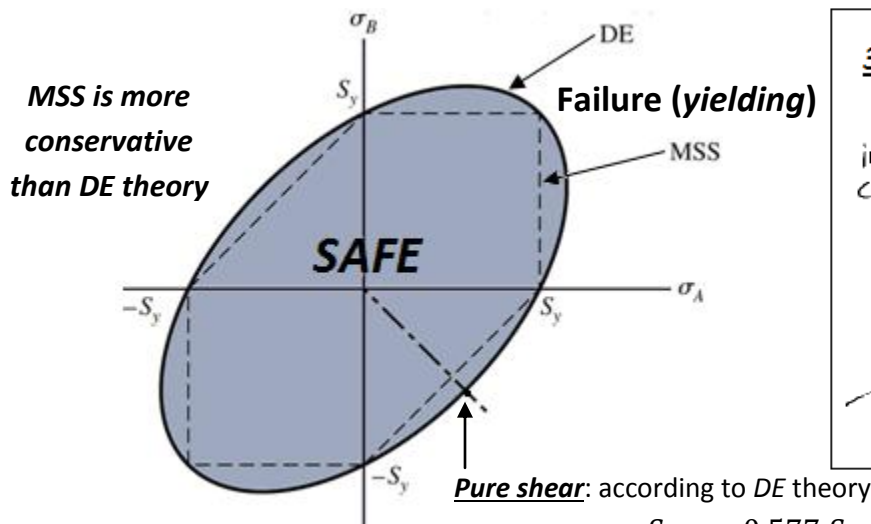
$$\sigma' \geq S_y$$

- For plane stress (σ_A & σ_B only), the *Von Mises stress* reduces to:

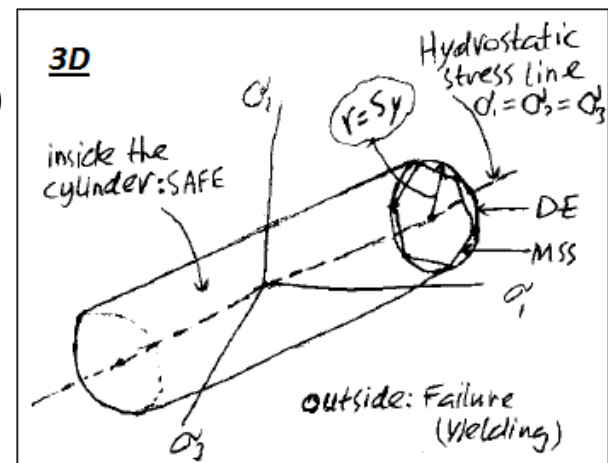
Plane Stress

$$\sigma' = \sqrt{\sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2}$$

- This equation describes a rotated ellipse in the σ_A, σ_B plane:



$$S_{ys} = 0.577 S_y \text{ (More accurate)}$$



- For any general state of stress, in x, y & z components, *Von Mises stress* can be found as:

- 3D:
$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

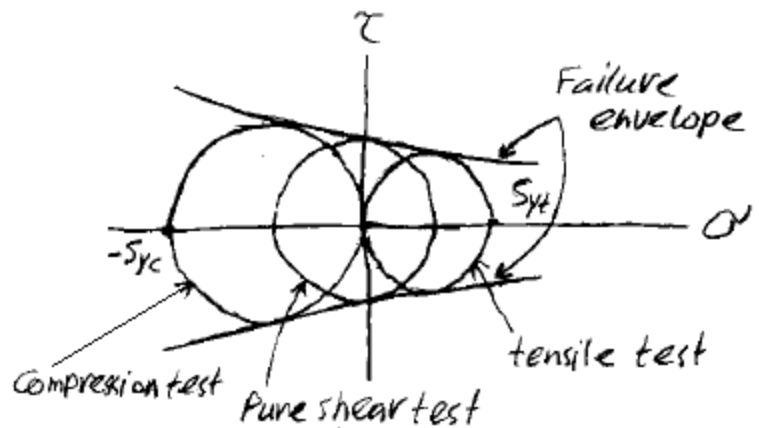
- Plane stress:
$$\sigma' = \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$

See Example 5-1 from text

Coulomb-Mohr Theory (for Ductile Materials)

The *MSS & DE* theories assume the tensile and compressive yield strength to be equal. However, some materials have compressive strength larger than tensile strength. The *Mohr & Coulomb-Mohr* theories are applicable for such materials.

- The *Mohr & Coulomb-Mohr* theories predict that failure will occur when the Mohr's circle (for any state of stress) “touches” a failure envelope (curve or line) which is defined by Mohr's circle for simple tension, shear and compression tests.
- Mohr's theory: the failure envelope is obtained from Mohr's circles obtained from three simple tension, shear and compression tests.
 - The failure envelope can be a straight-line or any other type of curve.
 - It is a graphical method.

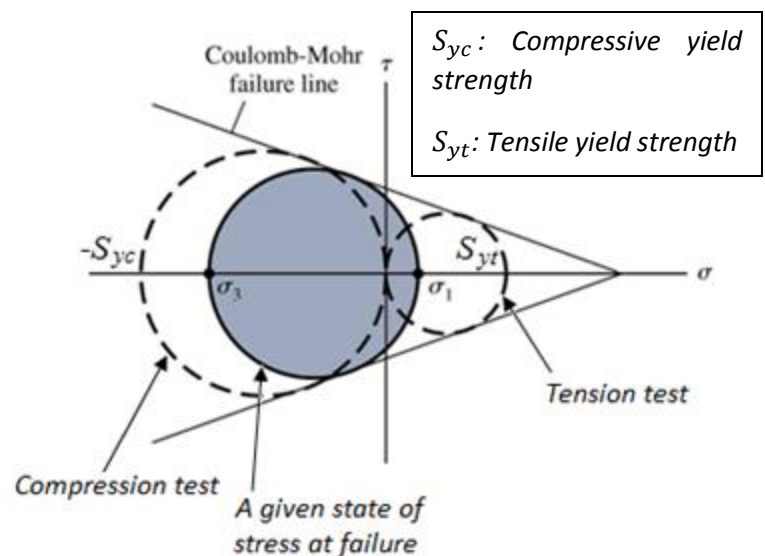


- Coulomb-Mohr theory: it assumes the failure envelope to be a straight line, thus shear test is not needed (*only tension and compression tests*).

- Using similar triangles we can obtain the failure condition to be:

$$\frac{\sigma_1}{S_{yt}} - \frac{\sigma_3}{S_{yc}} \geq 1 \quad (1)$$

- For plane stress (and $\sigma_A \geq \sigma_B$) three cases can be observed:

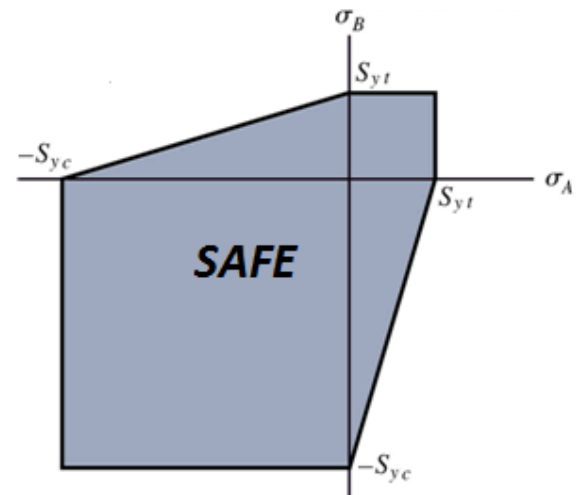


- **Case 1:** $\sigma_A \geq \sigma_B \geq 0$, thus $\sigma_1 = \sigma_A$ & $\sigma_3 = 0 \rightarrow \boxed{\sigma_A \geq S_{yt}}$
- **Case 2:** $\sigma_A \geq 0 \geq \sigma_B$, thus $\sigma_1 = \sigma_A$ & $\sigma_3 = \sigma_B \rightarrow \boxed{\frac{\sigma_A}{S_{yt}} - \frac{\sigma_B}{S_{yc}} \geq 1}$
- **Case 3:** $0 \geq \sigma_A \geq \sigma_B$, thus $\sigma_1 = 0$ & $\sigma_3 = \sigma_B \rightarrow \boxed{\sigma_B \leq -S_{yc}}$

- For pure shear ($\tau = \sigma_1 = -\sigma_3$) from equation (1), the shear yield strength can be found to be:

Shear yield strength
$$\boxed{S_{ys} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}}}$$

- Note that when the compressive and tensile strengths are equal ($S_{yt} = S_{yc} = S_y$), the *Coulomb-Mohr* theory turns to be the *MSS* theory.



Failure Theories for Brittle Materials

Brittle materials are classified such that $\epsilon_f < 0.05$ and they usually do not have identifiable yield strength. Thus, they are usually classified by the ultimate tensile (S_{ut}) and ultimate compressive (S_{uc}) strengths, which are used to define failure. It is typical for brittle materials to have the compressive strength larger than the tensile strength.

Maximum Normal Stress Theory (for Brittle Materials)

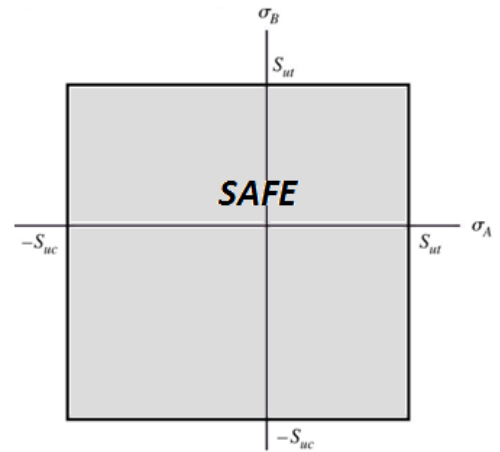
The *MNS* theory predicts that failure will occur when any of the three principal stresses equals or exceeds the ultimate strength obtained from tensile or compressive test.

- For general state of stress ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), according to *MNS* theory, failure occurs whenever:

$$\boxed{\sigma_1 \geq S_{ut} \quad \text{or} \quad \sigma_3 \leq -S_{uc}}$$

- For plane stress (σ_A & σ_B):

- $\sigma_A \geq \sigma_B \rightarrow \boxed{\sigma_A \geq S_{ut} \quad \text{or} \quad \sigma_B \leq -S_{uc}}$
- $\sigma_A \leq \sigma_B \rightarrow \boxed{\sigma_A \leq -S_{uc} \quad \text{or} \quad \sigma_B \geq S_{ut}}$



Coulomb-Mohr Theory and Modifications (for Brittle Materials)

The same Coulomb-Mohr theory used for ductile materials is also used for brittle materials but instead of using the yield strength, the ultimate strength is used to define failure.

- Brittle-Coulomb-Mohr: (Plane Stress) σ_A & σ_B

Failure occurs when:

- $\boxed{\sigma_A \geq S_{ut}}$ for $\sigma_A \geq \sigma_B \geq 0$
- $\boxed{\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} \geq 1}$ for $\sigma_A \geq 0 \geq \sigma_B$
- $\boxed{\sigma_B \leq -S_{uc}}$ for $0 \geq \sigma_A \geq \sigma_B$

- Modified I - Mohr:

Failure occurs when:

- $\boxed{\sigma_A \geq S_{ut}}$ for $\begin{cases} \sigma_A \geq \sigma_B \geq 0 \\ \text{or} \\ \sigma_A \geq 0 \geq \sigma_B \text{ and } \left| \frac{\sigma_B}{\sigma_A} \right| \leq 1 \end{cases}$
- $\boxed{\frac{(S_{uc} - S_{ut})\sigma_A}{S_{uc} S_{ut}} - \frac{\sigma_B}{S_{uc}} \geq 1}$ for $\sigma_A \geq 0 \geq \sigma_B \text{ and } \left| \frac{\sigma_B}{\sigma_A} \right| > 1$
- $\boxed{\sigma_B \leq -S_{uc}}$ for $0 \geq \sigma_A \geq \sigma_B$

Linear relation \rightarrow

- Modified II - Mohr:

Failure occurs when:

- $\sigma_A \geq S_{ut}$

Parabolic relation →

- $\frac{\sigma_A}{S_{ut}} + \left(\frac{\sigma_B + S_{ut}}{S_{ut} - S_{uc}} \right)^2 \geq 1$

- $\sigma_B \leq -S_{uc}$

$$\text{for } \begin{cases} \sigma_A \geq \sigma_B \geq 0 \\ \text{or} \\ \sigma_A \geq 0 \geq \sigma_B \text{ and } \left| \frac{\sigma_B}{\sigma_A} \right| \leq 1 \end{cases}$$

$$\text{for } \sigma_A \geq 0 \geq \sigma_B \text{ and } \left| \frac{\sigma_B}{\sigma_A} \right| > 1$$

$$\text{for } 0 \geq \sigma_A \geq \sigma_B$$

➤ Mod II - Mohr is the most accurate for loading in the fourth quadrant.

