MECHANICS OF MATERIALS

UNIAXIAL STRESS-STRAIN

Stress-Strain Curve for Mild Steel

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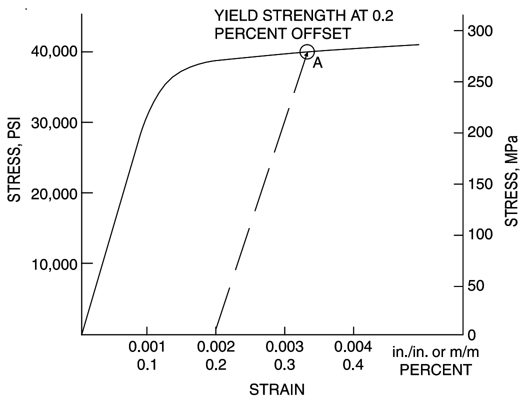

The slope of the linear portion of the curve equals the modulus of elasticity.

DEFINITIONS

Engineering Strain

$\varepsilon = \Delta L/L_o$, where

ε = engineering strain (units per unit),

ΔL = change in length (units) of member,

L_o = original length (units) of member.

Percent Elongation

$$\%\ \text{Elongation} = \left(\frac{\Delta L}{L_o}\right) \times 100$$

Percent Reduction in Area (RA)

The % reduction in area from initial area, A_i, to final area, A_f, is:

$$\%RA = \left(\frac{A_i - A_f}{A_i}\right) \times 100$$

Shear Stress-Strain

$\gamma = \tau/G$, where

γ = shear strain,

τ = shear stress, and

G = *shear modulus* (constant in linear torsion-rotation relationship).

$$G = \frac{E}{2(1+v)}, \text{ where}$$

E = modulus of elasticity

v = *Poisson's ratio*, and

= – (lateral strain)/(longitudinal strain).

Uniaxial Loading and Deformation

$\sigma = P/A$, where

σ = stress on the cross section,

P = loading, and

A = cross-sectional area.

$\varepsilon = \delta/L$, where

δ = elastic longitudinal deformation and

L = length of member.

$$E = \sigma/\varepsilon = \frac{P/A}{\delta/L}$$

$$\delta = \frac{PL}{AE}$$

True stress is load divided by actual cross-sectional area whereas engineering stress is load divided by the initial area.

THERMAL DEFORMATIONS

$\delta_t = \alpha L(T - T_o)$, where

δ_t = deformation caused by a change in temperature,

α = temperature coefficient of expansion,

L = length of member,

T = final temperature, and

T_o = initial temperature.

CYLINDRICAL PRESSURE VESSEL

Cylindrical Pressure Vessel

For internal pressure only, the stresses at the inside wall are:

$$\sigma_t = P_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_i$$

For external pressure only, the stresses at the outside wall are:

$$\sigma_t = -P_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_o, \text{ where}$$

σ_t = tangential (hoop) stress,

σ_r = radial stress,

P_i = internal pressure,

P_o = external pressure,

r_i = inside radius, and

r_o = outside radius.

For vessels with end caps, the axial stress is:

$$\sigma_a = P_i \frac{r_i^2}{r_o^2 - r_i^2}$$

σ_t, σ_r, and σ_a are principal stresses.

♦ Flinn, Richard A. & Paul K. Trojan, *Engineering Materials & Their Applications*, 4th ed., Houghton Mifflin Co., Boston, 1990.

When the thickness of the cylinder wall is about one-tenth or less of inside radius, the cylinder can be considered as thin-walled. In which case, the internal pressure is resisted by the hoop stress and the axial stress.

$$\sigma_t = \frac{P_i r}{t} \quad \text{and} \quad \sigma_a = \frac{P_i r}{2t}$$

where t = wall thickness.

STRESS AND STRAIN

Principal Stresses

For the special case of a *two-dimensional* stress state, the equations for principal stress reduce to

$$\sigma_a, \sigma_b = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_c = 0$$

The two nonzero values calculated from this equation are temporarily labeled σ_a and σ_b and the third value σ_c is always zero in this case. Depending on their values, the three roots are then labeled according to the convention:
algebraically largest = σ_1, *algebraically smallest* = σ_3, *other* = σ_2. A typical 2D stress element is shown below with all indicated components shown in their positive sense.

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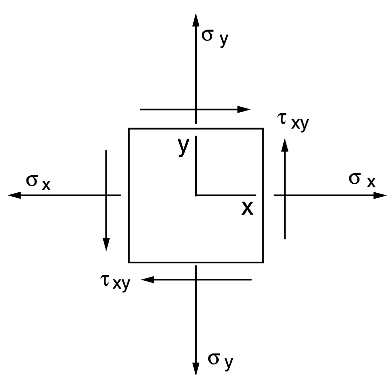

Mohr's Circle – Stress, 2D

To construct a Mohr's circle, the following sign conventions are used.

1. Tensile normal stress components are plotted on the horizontal axis and are considered positive. Compressive normal stress components are negative.
2. For constructing Mohr's circle only, shearing stresses are plotted above the normal stress axis when the pair of shearing stresses, acting on opposite and parallel faces of an element, forms a clockwise couple. Shearing stresses are plotted below the normal axis when the shear stresses form a counterclockwise couple.

The circle drawn with the center on the normal stress (horizontal) axis with center, C, and radius, R, where

$$C = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The two nonzero principal stresses are then:

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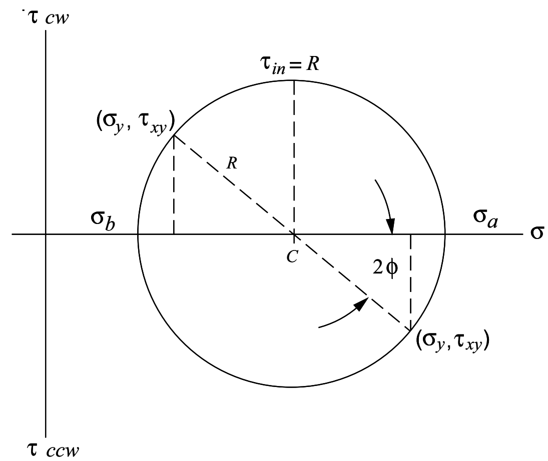
$\sigma_a = C + R$
$\sigma_b = C - R$



The maximum *inplane* shear stress is $\tau_{in} = R$. However, the maximum shear stress considering three dimensions is always

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Hooke's Law

Three-dimensional case:

$$\varepsilon_x = (1/E)[\sigma_x - v(\sigma_y + \sigma_z)] \qquad \gamma_{xy} = \tau_{xy}/G$$
$$\varepsilon_y = (1/E)[\sigma_y - v(\sigma_z + \sigma_x)] \qquad \gamma_{yz} = \tau_{yz}/G$$
$$\varepsilon_z = (1/E)[\sigma_z - v(\sigma_x + \sigma_y)] \qquad \gamma_{zx} = \tau_{zx}/G$$

Plane stress case ($\sigma_z = 0$):

$\varepsilon_x = (1/E)(\sigma_x - v\sigma_y)$
$\varepsilon_y = (1/E)(\sigma_y - v\sigma_x)$
$\varepsilon_z = -(1/E)(v\sigma_x + v\sigma_y)$

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \frac{E}{1 - v^2}\begin{bmatrix}1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{Bmatrix}$$

Uniaxial case ($\sigma_y = \sigma_z = 0$): $\sigma_x = E\varepsilon_x$ or $\sigma = E\varepsilon$, where

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = normal strain,

$\sigma_x, \sigma_y, \sigma_z$ = normal stress,

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ = shear strain,

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ = shear stress,

E = modulus of elasticity,

G = shear modulus, and

v = Poisson's ratio.

♦ Crandall, S.H. and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.

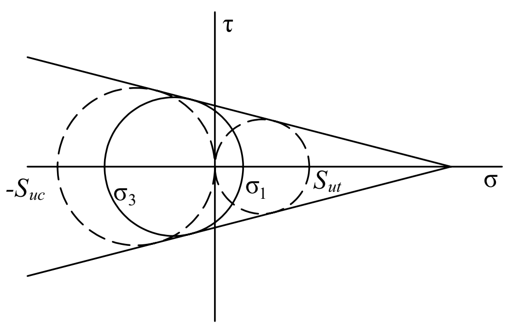
STATIC LOADING FAILURE THEORIES

See **MATERIALS SCIENCE/STRUCTURE OF MATTER** for Stress Concentration in Brittle Materials.

Brittle Materials

Maximum-Normal-Stress Theory
The maximum-normal-stress theory states that failure occurs when one of the three principal stresses equals the strength of the material. If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then the theory predicts that failure occurs whenever $\sigma_1 \geq S_{ut}$ or $\sigma_3 \leq -S_{uc}$ where S_{ut} and S_{uc} are the tensile and compressive strengths, respectively.

Coulomb-Mohr Theory
The Coulomb-Mohr theory is based upon the results of tensile and compression tests. On the σ, τ coordinate system, one circle is plotted for S_{ut} and one for S_{uc}. As shown in the figure, lines are then drawn tangent to these circles. The Coulomb-Mohr theory then states that fracture will occur for any stress situation that produces a circle that is either tangent to or crosses the envelope defined by the lines tangent to the S_{ut} and S_{uc} circles.



If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and $\sigma_3 < 0$, then the theory predicts that yielding will occur whenever

$$\frac{\sigma_1}{S_{ut}} - \frac{\sigma_3}{S_{uc}} \geq 1$$

Ductile Materials

Maximum-Shear-Stress Theory
The maximum-shear-stress theory states that yielding begins when the maximum shear stress equals the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield. If $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then the theory predicts that yielding will occur whenever $\tau_{max} \geq S_y/2$ where S_y is the yield strength.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Distortion-Energy Theory
The distortion-energy theory states that yielding begins whenever the distortion energy in a unit volume equals the distortion energy in the same volume when uniaxially stressed to the yield strength. The theory predicts that yielding will occur whenever

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}\right]^{1/2} \geq S_y$$

The term on the left side of the inequality is known as the effective or Von Mises stress. For a biaxial stress state the effective stress becomes

$$\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2}$$

or

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2}$$

where σ_A and σ_B are the two nonzero principal stresses and σ_x, σ_y, and τ_{xy} are the stresses in orthogonal directions.

VARIABLE LOADING FAILURE THEORIES

Modified Goodman Theory: The modified Goodman criterion states that a fatigue failure will occur whenever

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} \geq 1 \quad \text{or} \quad \frac{\sigma_{max}}{S_y} \geq 1, \quad \sigma_m \geq 0,$$

where

S_e = fatigue strength,

S_{ut} = ultimate strength,

S_y = yield strength,

σ_a = alternating stress, and

σ_m = mean stress.

$\sigma_{max} = \sigma_m + \sigma_a$

Soderberg Theory: The Soderberg theory states that a fatigue failure will occur whenever

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} \geq 1 \qquad \sigma_m \geq 0$$

Endurance Limit for Steels: When test data is unavailable, the endurance limit for steels may be estimated as

$$S'_e = \left\{\begin{matrix} 0.5\, S_{ut}, S_{ut} \leq 1,400 \text{ MPa} \\ 700 \text{ MPa}, S_{ut} > 1,400 \text{ MPa} \end{matrix}\right\}$$

Endurance Limit Modifying Factors: Endurance limit modifying factors are used to account for the differences between the endurance limit as determined from a rotating beam test, S'_e, and that which would result in the real part, S_e.

$$S_e = k_a k_b k_c k_d k_e S'_e$$

where

Surface Factor, $k_a = aS_{ut}^b$

Surface Finish	Factor *a*		Exponent *b*
	kpsi	**MPa**	
Ground	1.34	1.58	–0.085
Machined or CD	2.70	4.51	–0.265
Hot rolled	14.4	57.7	–0.718
As forged	39.9	272.0	–0.995

Size Factor, k_b:

For bending and torsion:

$d \le 8$ mm; $k_b = 1$

$8 \text{ mm} \le d \le 250$ mm; $k_b = 1.189 d_{eff}^{-0.097}$

$d > 250$ mm; $0.6 \le k_b \le 0.75$

For axial loading: $k_b = 1$

Load Factor, k_c:

$k_c = 0.923$ axial loading, $S_{ut} \le 1{,}520$ MPa

$k_c = 1$ axial loading, $S_{ut} > 1{,}520$ MPa

$k_c = 1$ bending

$k_c = 0.577$ torsion

Temperature Factor, k_d:

for T ≤ 450°C, $k_d = 1$

Miscellaneous Effects Factor, k_e: Used to account for strength reduction effects such as corrosion, plating, and residual stresses. In the absence of known effects, use $k_e = 1$.

TORSION

Torsion stress in circular solid or thick-walled (t > 0.1 *r*) shafts:

$$\tau = \frac{Tr}{J}$$

where J = polar moment of inertia (see table at end of **STATICS** section).

TORSIONAL STRAIN

$$\gamma_{\phi z} = \lim_{\Delta z \to 0} r(\Delta\phi/\Delta z) = r(d\phi/dz)$$

The shear strain varies in direct proportion to the radius, from zero strain at the center to the greatest strain at the outside of the shaft. $d\phi/dz$ is the twist per unit length or the rate of twist.

$$\tau_{\phi z} = G\gamma_{\phi z} = Gr(d\phi/dz)$$

$$T = G(d\phi/dz)\int_A r^2 dA = GJ(d\phi/dz)$$

$$\phi = \int_o^L \frac{T}{GJ}dz = \frac{TL}{GJ}, \text{ where}$$

ϕ = total angle (radians) of twist,

T = torque, and

L = length of shaft.

T/ϕ gives the *twisting moment per radian of twist*. This is called the *torsional stiffness* and is often denoted by the symbol *k* or *c*.

For Hollow, Thin-Walled Shafts

$$\tau = \frac{T}{2A_m t}, \text{ where}$$

t = thickness of shaft wall and

A_m = the total mean area enclosed by the shaft measured to the midpoint of the wall.

BEAMS

Shearing Force and Bending Moment Sign Conventions

1. The bending moment is *positive* if it produces bending of the beam *concave upward* (compression in top fibers and tension in bottom fibers).
2. The shearing force is *positive* if the *right portion of the beam tends to shear downward with respect to the left.*

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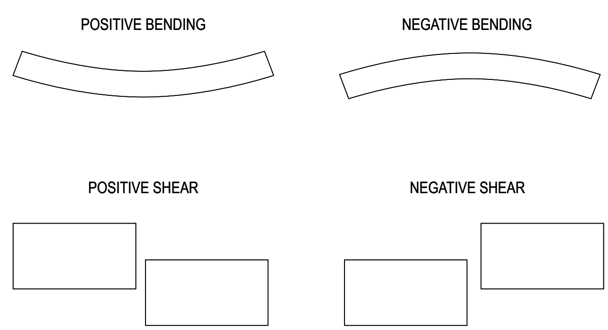


♦ Timoshenko, S. and Gleason H. MacCullough, *Elements of Strengths of Materials*, K. Van Nostrand Co./Wadsworth Publishing Co., 1949.

The relationship between the load (q), shear (V), and moment (M) equations are:

$$q(x) = -\frac{dV(x)}{dx}$$

$$V = \frac{dM(x)}{dx}$$

$$V_2 - V_1 = \int_{x_1}^{x_2} \left[-q(x)\right] dx$$

$$M_2 - M_1 = \int_{x_1}^{x_2} V(x)\, dx$$

Stresses in Beams

$\varepsilon_x = -y/\rho$, where

ρ = the radius of curvature of the deflected axis of the beam, and

y = the distance from the neutral axis to the longitudinal fiber in question.

Using the stress-strain relationship $\sigma = E\varepsilon$,
Axial Stress: $\sigma_x = -Ey/\rho$, where

σ_x = the normal stress of the fiber located y-distance from the neutral axis.

$1/\rho = M/(EI)$, where

M = the moment at the section and

I = the *moment of inertia* of the cross section.

$\sigma_x = -My/I$, where

y = the distance from the neutral axis to the fiber location above or below the axis. Let $y = c$, where c = distance from the neutral axis to the outermost fiber of a symmetrical beam section.

$\sigma_x = \pm Mc/I$

Let $S = I/c$: then, $\sigma_x = \pm M/S$, where

S = the *elastic section modulus* of the beam member.

Transverse shear flow: $q = VQ/I$ and

Transverse shear stress: $\tau_{xy} = VQ/(Ib)$, where

q = shear flow,

τ_{xy} = shear stress on the surface,

V = shear force at the section,

b = width or thickness of the cross-section, and

Q = $A'\overline{y'}$, where

A' = area above the layer (or plane) upon which the desired transverse shear stress acts and

$\overline{y'}$ = distance from neutral axis to area centroid.

Deflection of Beams

Using $1/\rho = M/(EI)$,

$$EI\frac{d^2y}{dx^2} = M, \text{ differential equation of deflection curve}$$

$$EI\frac{d^3y}{dx^3} = dM(x)/dx = V$$

$$EI\frac{d^4y}{dx^4} = dV(x)/dx = -q$$

Determine the deflection curve equation by double integration (apply boundary conditions applicable to the deflection and/or slope).

$$EI\,(dy/dx) = \int M(x)\,dx$$

$$EIy = \int\left[\int M(x)\,dx\right] dx$$

The constants of integration can be determined from the physical geometry of the beam.

COLUMNS

For long columns with pinned ends:
Euler's Formula

$$P_{cr} = \frac{\pi^2 EI}{\ell^2}, \text{ where}$$

P_{cr} = critical axial loading,

ℓ = unbraced column length.

substitute $I = r^2 A$:

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(\ell/r)^2}, \text{ where}$$

r = *radius of gyration* and

ℓ/r = *slenderness ratio* for the column.

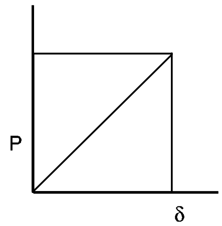
For further column design theory, see the **CIVIL ENGINEERING** and **MECHANICAL ENGINEERING** sections.

ELASTIC STRAIN ENERGY

If the strain remains within the elastic limit, the work done during deflection (extension) of a member will be transformed into potential energy and can be recovered.

If the final load is P and the corresponding elongation of a tension member is δ, then the total energy U stored is equal to the work W done during loading.

$$U = W = P\delta/2$$

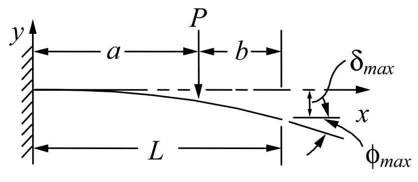
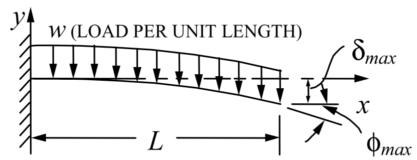
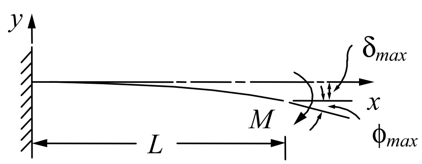
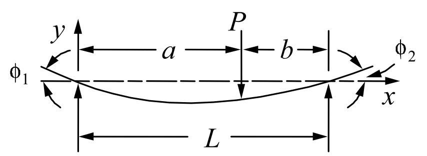
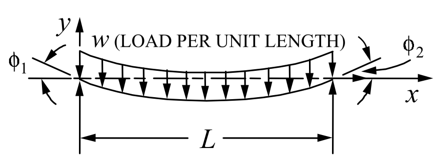
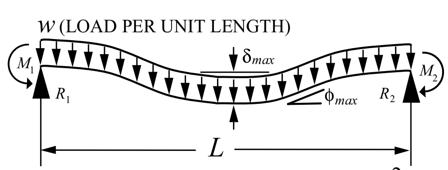


The strain energy per unit volume is

$$u = U/AL = \sigma^2/2E \qquad \text{(for tension)}$$

MATERIAL PROPERTIES

Material		Units	Steel	Aluminum	Cast Iron	Wood (Fir)
Modulus of Elasticity, E		Mpsi	29.0	10.0	14.5	1.6
		GPa	200.0	69.0	100.0	11.0
Modulus of Rigidity, G		Mpsi	11.5	3.8	6.0	0.6
		GPa	80.0	26.0	41.4	4.1
Poisson's Ratio, v			0.30	0.33	0.21	0.33
Coefficient of Thermal Expansion, α		$10^{-6}/°F$	6.5	13.1	6.7	1.7
		$10^{-6}/°C$	11.7	23.6	12.1	3.0

Beam Deflection Formulas – Special Cases (δ is positive downward)			
	$\delta = \frac{Pa^2}{6EI}(3x - a)$, for $x > a$ $\delta = \frac{Px^2}{6EI}(-x + 3a)$, for $x \le a$	$\delta_{max} = \frac{Pa^2}{6EI}(3L - a)$	$\phi_{max} = \frac{Pa^2}{2EI}$
	$\delta = \frac{wx^2}{24EI}\left(x^2 + 6L^2 - 4Lx\right)$	$\delta_{max} = \frac{wL^4}{8EI}$	$\phi_{max} = \frac{wL^3}{6EI}$
	$\delta = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{ML^2}{2EI}$	$\phi_{max} = \frac{ML}{EI}$
 $R_1 = Pb/L$ $R_2 = Pa/L$	$\delta = \frac{Pb}{6LEI}\left[\frac{L}{b}(x-a)^3 - x^3 + \left(L^2 - b^2\right)x\right]$, for $x > a$ $\delta = \frac{Pb}{6LEI}\left[-x^3 + \left(L^2 - b^2\right)x\right]$, for $x \le a$	$\delta_{max} = \frac{Pb\left(L^2 - b^2\right)^{3/2}}{9\sqrt{3}LEI}$ at $x = \sqrt{\frac{L^2 - b^2}{3}}$	$\phi_1 = \frac{Pab(2L - a)}{6LEI}$ $\phi_2 = \frac{Pab(2L - b)}{6LEI}$
 $R_1 = wL/2$ $R_2 = wL/2$	$\delta = \frac{wx}{24EI}\left(L^3 - 2Lx^2 + x^3\right)$	$\delta_{max} = \frac{5wL^4}{384EI}$	$\phi_1 = \phi_2 = \frac{wL^3}{24EI}$
 $R_1 = R_2 = \frac{wL}{2}$ and $M_1 = M_2 = \frac{wL^2}{12}$	$\delta(x) = \frac{wx^2}{24EI}(L^2 - Lx + x^2)$	$\lvert\delta_{max}\rvert = \frac{wL^4}{384EI}$ at $x = \frac{L}{2}$	$\lvert\phi_{max}\rvert = 0.008\frac{wL^3}{24EI}$ at $x = \frac{1}{2} \pm \frac{L}{\sqrt{12}}$

Crandall, S.H. and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.